minutes. Figure 2 shows a general shape of the free surface obtained by computing the wave profile along nine arbitrary sections. It shows the contour lines spaced at $0.04v^{-1}$ intervals and we see from it that the wave pattern behind a moving ellipsoid is much more complex than one would expect from the asymptotic theory.

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CONVECTIVE INSTABILITY OF A FLUID LAYER IN A MODULATED EXTERNAL FORCE FIELD

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The onset of convection in a layer of an incompressible fluid with free boundaries is considered. The temperature at the layer boundaries, the density of the internal heat sources and the strength of the gravity field are all assumed to be T -periodic. The existence of the critical Rayleigh number and the T-periodicity of the neutral perturbation are proved for the case when the unperturbed temperature gradient is negative throughout the layer. These results are obtained by reducing the linearized problem to an ordinary differential equation in certain Banach space and applying the theory of the linear positive operators [1].

The onset of convection under the action of time-periodic forces is dealt with in [2 - 9]. The stability of equilibrium of a horizontal layer with free and rigid boundaries was investigated and numerical methods were used to determine the limits of stability [2] under the assumption that the temperature gradient was independent of the vertical coordinate. The method of averaging over small oscillations was used in [3, 4] to study the influence of high frequency vertical oscillations on the onset of convection. Use of the method of averaging for the abstract parabolic equations, in particular for the convection problem, was substantiated in [5, 6]. Convection in a cavity of square cross section in the case when the fluid is heated from below and is acted upon by vibrational forces, is studied in [7] and a numerical solution given for the nonlinear convection equations. The stability of equilibrium in the case when the temperature gradient depends on the vertical coordinate is studied in [8, 9], namely the convection in a deep vessel the surface temperature in which varies periodically with time is dealt with in [8], and the convection in a horizontal layer with the temperature varying periodically at the free boundaries, the amplitude of the modulations being small, in [9].

In the present paper we consider the onset of convection in a horizontal layer of a viscous incompressible fluid bounded by the surfaces $z = 0$ and h. The density of distribution of the heat sources within the layer is $aF(z, t)$. The temperature at the horizontal boundaries is given and varies as $a\varphi(z, t)$. The fluid layer executes vertical oscillations with the acceleration equal to $g[\Phi(t) - 1]$.

We assume the functions F, Φ and φ to be smooth and T-periodic in t. We also assume that the relative velocity of motion of the fluid $\mathbf{v}'$ and the temperature θ' are periodic in x and y their respective periods equal to $2\pi/\alpha_1$ and $2\pi/\alpha_2$ and that the fluid layer cannot be displaced as a whole along the x, y-plane.

$$\int_{-\pi/\alpha_2}^{\pi/\alpha_2}\int_0^1 v_x' dydz = \int_{-\pi/\alpha_1}^{\pi/\alpha_1}\int_0^1 v_y' dxdz = 0$$

We further consider the stability of the state of rest during which

$$\mathbf{v}_0' = 0, \quad \theta_0' = a\theta_0(z, t), \quad P_0' = \beta g a \Phi(t)\int_0^z \theta_0 dz + \psi(t) \tag{1}$$

where $\psi(t)$ is an arbitrary function of time. The function $\psi(t)$ can be determined uniquely, it tne value of the pressure is known at any one point for all t.

The equilibrium temperature θ_0 can be found from the following problem:

$$\frac{\partial\theta}{\partial t} = \chi\frac{\partial^2\theta}{\partial z^2} + F(z, t)$$
$$\theta|_{z=0} = \varphi(0, t), \quad \theta|_{z=1} = \varphi(1, t), \quad \theta(z, t) = \theta(z, t+T)$$

where we assume that F and φ are infinitely differentiable functions. In this case $\theta_0(z, t)$ exhibits the same property.

In the dimensionless coordinates, the linearized equations describing small perturbations of equilibrium assume the following form

$$\frac{1}{\sqrt{p}}\frac{\partial\mathbf{v}}{\partial t} = -\nabla P + \Delta\mathbf{v} + \mathbf{j}R\Phi(t)\theta$$
$$\sqrt{p}\frac{\partial\theta}{\partial t} = \Delta\theta + \mathbf{j}Rc(z, t)\mathbf{v}, \qquad \operatorname{div}\mathbf{v} = 0 \tag{2}$$
$$R^2 = \mathrm{Ra} = \frac{g\beta a h^4}{\nu\chi}, \qquad p = \frac{\nu}{\chi}, \qquad c(z, t) = -\frac{\partial\theta_0}{\partial z}$$

where Ra and p are the Rayleigh and Prandtl numbers and **j** is the unit vector directed vertically upwards. The conditions for the corresponding integrals to be periodic in x and y and to be equal to zero, remain unchanged. We seek the solution of (2) in the form

$$v_x = v_1(z, t)\sin k_1\alpha_1 x \cos k_2\alpha_2 y \tag{3}$$

$$v_y = v_2(z, t)\cos k_1\alpha_1 x \sin k_2\alpha_2 y$$

$$(v_z, \theta, P) = (w(z, t), \tau(z, t), q(z, t))\cos k_1\alpha_1 x \cos k_2\alpha_2 y$$

where k_1 and k_2 are natural numbers.

Inserting (3) into (2) and eliminating the velocity components v_1 and v_2 and the pressure q, we obtain the following system of equations for w and τ :

$$\left(\frac{\partial}{\partial t} - \sqrt{p}L\right) Lw = -R\alpha^2 \sqrt{p}\Phi(t)\tau, \quad \left(\frac{\partial}{\partial t} - \frac{1}{\sqrt{p}}L\right)\tau = R\frac{c(z, t)}{\sqrt{p}} w \tag{4}$$

$$L = \partial^2/\partial z^2 - \alpha^2, \qquad \alpha^2 = (k_1\alpha_1)^2 + (k_2\alpha_2)^2$$

The boundary conditions at the free edge of the layer are

$$w = \partial^2 w / \partial z^2 = 0, \qquad \tau = 0 \qquad (z = 0,1) \tag{5}$$

Theorem: Let $\Phi(t) > 0$ and $c(z, t) > 0$ for all $z \in [0, 1]$, $t > 0$. Then a critical Rayleigh number Ra_* exists such that when $R > \sqrt{\mathrm{Ra}_*}$ the state of rest (1) is unstable and the problem (4) and (5) has a nonzero solution

$$w = e^{\sigma t}\tilde{w}(z, t), \qquad \tau = e^{\sigma t}\tilde{\tau}(z, t) \tag{6}$$

where $\sigma > 0$ and the functions $\tilde{w}$ and $\tilde{\tau}$ are T-periodic in t .

Proof. Let G be the Green operator of the differential operator L with the boundary conditions $w(0) = w(1) = 0$. Using the operator G we obtain

$$\left(\frac{\partial}{\partial t} - \sqrt{p}L\right) w = R\alpha^2 \sqrt{p}\Phi(t)G\tau, \quad \left(\frac{\partial}{\partial t} - \frac{1}{\sqrt{p}}L\right)\tau = R\frac{c(z, t)}{\sqrt{p}} w$$

$$w\,|_{z=0,\,1} = \tau\,|_{z=0,\,1} = 0 \tag{7}$$

Let us introduce the space $E = L_p(0, 1) + L_p(0, 1)$, $p \geqslant 1$ consisting of a simple sum of two spaces L_p. The pair $\xi = (w, \tau)$ represents an element of space E and its norm is given by

$$\|\xi\|_E = \|w\|_{L_p} + \|\tau\|_{L_p}, \qquad \xi = (w, \tau)$$

We shall treat the system (7) as a differential equation in E

$$\frac{d\xi}{dt} = N\xi, \qquad N = \left\|\begin{matrix} \sqrt{p}L & R\alpha^2\sqrt{p}\Phi G \\ R\dfrac{c}{\sqrt{p}} & \dfrac{1}{\sqrt{p}}L \end{matrix}\right\| \tag{8}$$

Let us find the displacement operator $U(t)$ along the trajectories of the differential equation (8) over the time $0 \leqslant t \leqslant T$ [10], assuming that $\xi(t) = U(t)\xi(0)$. We shall consider a cone K of the following nonnegative vector functions in E : $\xi = (w, \tau) \in K$ when and only when $w(z)$ and $\tau(z) \geqslant 0$ for $z \in [0, 1]$. Then the cone K is reproducing and normal [1]. The operator $U(t)$ is positive with respect to K . This follows from the maximum principle for a second order parabolic equation [11] and from the positiveness of the operator G. We shall show that the monodromy operator $U_T = U(T)$ has in K an eigenvector with a corresponding positive and simple eigenvalue ρ.

First we consider the following homogeneous equation

$$\frac{\partial f}{\partial t}-\frac{1}{\sqrt{p}}Lf=0, \qquad f\,|_{z=0,\,1}=0$$

Its solution satisfying the initial condition $f(z,\ 0)=f_0\in D(L)$ is [12]

$$f(t)=V(t)f_0$$

where $V(t)$ is a semigroup of linear bounded operators in $L_p\,(0,\ 1)$, strongly continuous for $t>0$.

The problem (8) can be written as

$$\xi(t)=\zeta+R(B\xi)(t), \qquad \xi=(w,\ \tau) \tag{9}$$

$$\zeta=A(t)\,\xi_0=\begin{Vmatrix} V(pt)\,w_0 \\ V(t)\,\tau_0 \end{Vmatrix}, \quad (B\xi)(t)=\begin{Vmatrix} \alpha^2\sqrt{p}\int\limits_0^{pt} V(pt-s)\,\Phi(s)\,G\tau(s)\,ds \\ \frac{1}{\sqrt{p}}\int\limits_0^{t} V(t-s)\,c(z,\ s)\,w(s)\,ds \end{Vmatrix}$$

where $\xi_0=(w_0,\ \tau_0)$ denotes the initial value. The integrals in (9) should be understood as the limits on the norm of the corresponding Riemannian integral sums.

We write the monodromy operator U_T in the form

$$U_T\xi_0=\sum_{k=0}^{\infty}R^kC_k\xi_0=\sum_{k=0}^{\infty}R^kB^k\zeta\,|_{t=T} \tag{10}$$

The series in (10) converges uniformly on any sphere in E. We shall show that the operator U_T is completely continuous in E by establishing that every term in (10) is a completely continuous operator. The operator $V(t)$ is completely continuous in L_p for any fixed $t>0$, consequently the operator $C_0=A(T)$ is completely continuous in E.

Let us consider the operators B_ε

$$B_\varepsilon=\int\limits_0^{T-\varepsilon} V(T-s)\,A(s)\,ds$$

The operators B_ε are completely continuous in E. This follows from the complete continuity, for any fixed s of the operator appearing under the integral sign and from the convergence on the norm of the Riemannian integral sums [13]. When $\varepsilon\to 0$, the operators B_ε converge uniformly to the operator C_1, and this implies the complete continuity of the latter.

Complete continuity of the operator C_2 follows from the complete continuity of G in L_p and from the equation

$$C_2\xi_0=(\alpha^2GH_1w_0,\ \alpha^2GH_2\tau_0)$$

where H_1 and H_2 are bounded operators.

The complete continuity of the operators $C_k(k=3,4,\ldots)$ is proved similarly. From (10) we can now deduce that the operator U_T is completely continuous. We shall show that this operator is η_0-positive with respect to the cone K when $\eta_0=(\varphi_0,\ \varphi_0)$ and $\varphi_0(z)=\sin\pi z$ is the eigenfunction of the operator G corresponding to the smallest eigenvalue $\lambda=\pi^2+\alpha^2$.

As we know, the operator $V(t)$ is φ_0-positive with respect to the cone K_0 of nonnegative functions, and the following relations hold:

$$b(u,\ t)\varphi_0 \leqslant V(t)u \leqslant d(u,\ t)\varphi_0 \quad (u \in K_0,\ b,\ d > 0) \tag{11}$$

By the condition of the theorem constants m_1, m_2, n_1, n_2 exist such, that the following inequalities hold

$$m_1 \leqslant \Phi \leqslant m_2, \quad n_1 \leqslant c \leqslant n_2, \quad z \in [0, 1], \quad t \in [0, T] \tag{12}$$

To show that the operator U_T is η_0-bounded from below we note that for any $\xi_0 = (w_0,\ \tau_0) \in K$ we have

$$U_T\xi_0 \geqslant RB\zeta\,|_{t=T} = \tag{13}$$
$$= R\left(\alpha^2 \sqrt{p} \int\limits_0^{pT} V(pT - s)\,\Phi(s)\,GV(s)\,\tau_0 ds,\ \frac{1}{\sqrt{p}} \int\limits_0^T V(T - s)\,c(s)\,V(ps)\,w_0 ds\right)$$

This follows from (10) and from the positiveness of the operators A and B. The relation

$$V(t)\,\varphi_0 = \varphi_0 \exp\left(-\frac{\lambda}{\sqrt{p}}\,t\right) \tag{14}$$

can be checked directly.

From (11) - (14) we can derive the following estimate

$$U_T\xi_0 \geqslant RM_1(\xi_0)\,\eta_0, \qquad \eta_0 = (\varphi_0,\ \varphi_0)$$

$$M_1(\xi_0) = \min\left\{\frac{\alpha^2 m_1 \sqrt{p}}{\lambda} \int\limits_0^T e^{-\lambda \sqrt{p}\,(T-s)}\,b(\tau_0,\ ps)\,ds,\ \frac{n_1}{\sqrt{p}} \int\limits_0^T e^{-\lambda/\sqrt{p}\,(T-s)}\,b(w_0,\ s)\,ds\right\}$$

We can prove that the operator U_T is η_0-bounded from above by obtaining from (9), (11) and (14)

$$U_T\xi_0 \leqslant M_2(\xi_0)\,\eta_0$$

$$M_2(\xi_0) = R\gamma\delta e^{\gamma RT}, \quad \delta = \max\left\{\int\limits_0^T d(\tau_0,\ ps)\,ds,\ \int\limits_0^T d(w_0,\ s)\,ds\right\}$$

$$\gamma = \max\{\alpha^2 m_2 \sqrt{p},\quad n_2 / \sqrt{p}\}$$

Thus we have shown that the operator U_T is completely continuous and η_0-positive with respect to the reproducing cone K. Theorems of [1] imply that U_T has a unique eigenvector $\xi_0' = (w_0',\ \tau_0')$ in the cone K

$$U_T\xi_0' = \rho\xi_0', \qquad \rho \geqslant RM_1(\eta_0) > 0 \tag{15}$$

The positive multiplier ρ is simple and its absolute value exceeds those of the remaining eigenvalues.

The estimate (15) implies that when $R > R_* = 1 / M_1(\eta_0)$ the factor $\rho > 1$. The solution (6) can be obtained by setting $\xi = U(t)\xi_0'$. We note that for $t > 0$ the solution (6) is infinitely differentiable. Indeed, when $t > 0$, the functions $\Phi(t)$, $c(z,\ t)$, $V(t)\ w_0$ and $V(t)\tau_0$ are infinitely differentiable. The infinite differentiability of (6) now follows from (10) as the operator G acts from L_p into $W_p{}^2$ and from $W_p^{(l)}$ into $W_p^{(l+2)}$ for any $p \geqslant 1$. This completes the proof of the Theorem.

It was shown before that the multiplier ρ of maximum absolute value is positive. From this an analog of the variation of stability principle follows: the value $R_*(\alpha^2) = 1 / M_1(\eta_0)$ has a corresponding T-periodic solution (6) with $\sigma = 0$.

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SOME CONTACT PROBLEMS FOR AN ELASTIC INFINITE WEDGE

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Non-self-balanced homogeneous solutions of the mixed plane problem of elasticity theory for and infinite wedge $-\alpha \leqslant \theta \leqslant \alpha$, $0 \leqslant r < \infty$, one part of whose boundary $\theta = \pm\alpha$, $0 \leqslant r \leqslant 1$ is under the conditions of sliding constraint and the other is stress-free, are constructed and investigated. The solutions are of independent interest since they determined the state of stress of a wedge (a wedgelike strip in an elastic plane) on which a load equivalent to a longitudinal force P, a transverse force Q and a moment M (see Fig. 1) acts through a stiff yoke (a wedgelike stamp). Together with the statically balanced homogeneous solutions, they form a system of functions needed to solve mixed problems for elastic finite sectorial domains by the method elucidated in [1].

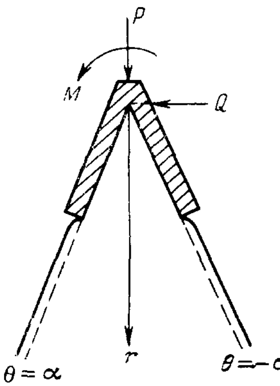

Fig. 1.

1. Symmetric problem. Let us write the condition on the wedge boundary for $\theta = \pm\alpha$:

$$u_{\theta} = 0 \quad \text{for} \quad 0 \leqslant r \leqslant 1, \qquad \sigma_\theta = 0 \quad \text{for} \quad 1 < r < \infty \tag{1.1}$$

$$\tau_{r\theta} = 0 \quad \text{for} \quad 0 \leqslant r < \infty \tag{1.2}$$

$$\sigma_\theta \sim (1-r)^{\varepsilon-1} \quad \text{for} \quad r \to 1-0 \qquad (\varepsilon > 0) \tag{1.3}$$